

FACTORIZING DERIVATIVES OF FUNCTIONS IN THE NEVANLINNA AND SMIRNOV CLASSES

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ABSTRACT. We prove that, given a function f in the Nevanlinna class \mathcal{N} and a positive integer n , there exist $g \in \mathcal{N}$ and $h \in \text{BMOA}$ such that $f^{(n)} = gh^{(n)}$. We may choose g to be zero-free, so it follows that the zero sets for the class $\mathcal{N}^{(n)} := \{f^{(n)} : f \in \mathcal{N}\}$ are the same as those for $\text{BMOA}^{(n)}$. Furthermore, while the set of all products $gh^{(n)}$ (with g and h as above) is strictly larger than $\mathcal{N}^{(n)}$, we show that the gap is not too large, at least when $n = 1$. Precisely speaking, the class $\{gh' : g \in \mathcal{N}, h \in \text{BMOA}\}$ turns out to be the smallest ideal space containing $\{f' : f \in \mathcal{N}\}$, where “ideal” means invariant under multiplication by H^∞ functions. Similar results are established for the Smirnov class \mathcal{N}^+ .

1. INTRODUCTION AND RESULTS

Let $\mathcal{H}(\mathbb{D})$ stand for the set of holomorphic functions on the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Given a class $X \subset \mathcal{H}(\mathbb{D})$ and an integer $n \in \mathbb{N} := \{1, 2, \dots\}$, we write

$$X^{(n)} := \{f^{(n)} : f \in X\},$$

where $f^{(n)}$ is the n th derivative of f . When $n = 1$, we also use the notation X' instead of $X^{(1)}$. Further, we denote by $\mathcal{Z}(X)$ the collection of zero sets for X ; a (discrete) subset E of \mathbb{D} will thus belong to $\mathcal{Z}(X)$ if and only if $E = \{z \in \mathbb{D} : f(z) = 0\}$ for some non-null function $f \in X$. Now, if X and Y are subclasses of $\mathcal{H}(\mathbb{D})$, we put

$$X \cdot Y := \{fg : f \in X, g \in Y\}.$$

Finally, a vector space X contained in $\mathcal{H}(\mathbb{D})$ is said to be *ideal* if

$$H^\infty \cdot X \subset X,$$

where, as usual, H^∞ is the space of bounded holomorphic functions on \mathbb{D} .

Our starting point is a result of Cohn and Verbitsky [3] which asserts, or rather implies, that

$$(1.1) \quad (H^p)^{(n)} = H^p \cdot \text{BMOA}^{(n)}$$

whenever $0 < p < \infty$ and $n \in \mathbb{N}$. Here, we write H^p for the classical (holomorphic) *Hardy spaces* on the disk, and BMOA for the “analytic subspace” of $\text{BMO} = \text{BMO}(\mathbb{T})$, the space of functions with *bounded mean oscillation* on the

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circle $\mathbb{T} := \partial\mathbb{D}$. More precisely, BMOA can be defined as $H^1 \cap \text{BMO}$; as to the definitions of (and background information on) H^p and BMO, the reader will find these standard matters in [5, Chapters II and VI].

For $n = 1$, identity (1.1) appeared in Cohn's earlier paper [2]. On the other hand, [3] extends (1.1) to the case of a fractional derivative and still further; indeed, more general factorization theorems involving tent spaces – and Triebel spaces – are actually established there. It is also shown in [3] that, when factoring $f^{(n)}$ (for $f \in H^p$) in the sense of (1.1), one may choose the H^p factor on the right to be an outer function. As a consequence, one sees that

$$(1.2) \quad \mathcal{Z}\left((H^p)^{(n)}\right) = \mathcal{Z}\left(\text{BMOA}^{(n)}\right).$$

In particular, for any fixed n , the zero sets for $(H^p)^{(n)}$ are the same for all $p \in (0, \infty)$. This last fact was contrasted in [3] with the Bergman space situation (where, by [7], different A^p spaces have different zero sets). We wish to add, in this connection, that a similar Bergman-type phenomenon (different zero sets for different p 's) is also encountered in certain “small” H^p -related spaces; namely, it occurs [4] for the *star-invariant subspaces* $H^p \cap \theta \overline{H_0^p}$ associated with an inner function θ .

Also related to (1.1) is Aleksandrov and Peller's work [1] that dealt with the case of $p \in [1, \infty)$ and $n = 1$. There, for a given $f \in H^p$, a *weak factorization* $f' = \sum_{j=1}^4 g_j h_j'$ was constructed with suitable $g_j \in H^p$ and $h_j \in H^\infty$. Yet another weak factorization theorem from [1], which establishes a connection between BMOA' and $(H^\infty)'$, will be employed in Section 4 below.

The purpose of this paper is to find out whether – and/or to which extent – the (strong) factorization theorem (1.1) carries over to the *Nevanlinna class* \mathcal{N} , or the *Smirnov class* \mathcal{N}^+ , in place of H^p .

Let us recall that \mathcal{N} is defined as the set of functions $f \in \mathcal{H}(\mathbb{D})$ with

$$\sup_{0 < r < 1} \int_{\mathbb{T}} \log^+ |f(r\zeta)| |d\zeta| < \infty,$$

while \mathcal{N}^+ is formed by those $f \in \mathcal{N}$ which satisfy

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{T}} \log^+ |f(r\zeta)| |d\zeta| = \int_{\mathbb{T}} \log^+ |f(\zeta)| |d\zeta|.$$

Equivalently, the elements of \mathcal{N} (resp., \mathcal{N}^+) are precisely the ratios u/v , with $u, v \in H^\infty$ and with v nonvanishing (resp., outer) on \mathbb{D} ; for this and other characterizations of the two classes, see [5, Chapter II].

As far as factorization theorems of the form (1.1) are concerned, we can hardly expect the behavior of \mathcal{N} or \mathcal{N}^+ to mimic that of H^p too closely. In fact, as we shall soon explain, it is the “easy” part of (1.1), i.e., the inclusion

$$(1.3) \quad (H^p)^{(n)} \supset H^p \cdot \text{BMOA}^{(n)}$$

that admits no extension to the Nevanlinna or Smirnov setting. Meanwhile, we remark that (1.3) is indeed easy to deduce, at least for $p = 2$, from the (not so

easy, but readily available) descriptions of $(H^p)^{(n)}$ and $\text{BMOA}^{(n)}$ as the appropriate Triebel spaces; see [11]. One of these tells us that, for $\varphi \in \mathcal{H}(\mathbb{D})$,

$$\varphi \in (H^p)^{(n)} \iff \int_{\mathbb{T}} \left(\int_0^1 |\varphi(r\zeta)|^2 (1-r)^{2n-1} dr \right)^{p/2} |d\zeta| < \infty$$

for all $n \in \mathbb{N}$ and $0 < p < \infty$, a fact that has no counterpart for \mathcal{N} or \mathcal{N}^+ . The other, which involves a Carleson measure characterization of BMOA , will be mentioned in Section 2 below.

Now, to see that the \mathcal{N} and \mathcal{N}^+ versions of (1.3) actually break down, already for $n = 1$, we recall Hayman's and Yanagihara's results from [6, 12] saying that neither \mathcal{N} nor \mathcal{N}^+ is invariant with respect to integration. More explicitly, Hayman [6] gave an example of a function $f \in \mathcal{N}$ whose antiderivative $F(z) := \int_0^z f(\zeta) d\zeta$ is not in \mathcal{N} ; Yanagihara [12] then strengthened this by showing that F need not be in \mathcal{N} even for $f \in \mathcal{N}^+$. Consequently, \mathcal{N}^+ is not contained in \mathcal{N}' , whence *a fortiori*

$$(1.4) \quad \mathcal{N} \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \not\subset (\mathcal{N}^+)'.$$

Since $\mathcal{N} \cdot \text{BMOA}'$ (resp., $\mathcal{N}^+ \cdot \text{BMOA}'$) contains \mathcal{N} (resp., \mathcal{N}^+), we readily deduce from (1.4) that

$$(1.5) \quad \mathcal{N} \cdot \text{BMOA}' \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \cdot \text{BMOA}' \not\subset (\mathcal{N}^+)'.$$

A similar conclusion holds for higher order derivatives as well.

We prove, however, that the “difficult” part of (1.1), i.e., the inclusion

$$(1.6) \quad (H^p)^{(n)} \subset H^p \cdot \text{BMOA}^{(n)}$$

does remain valid with either \mathcal{N} or \mathcal{N}^+ in place of H^p .

Theorem 1.1. *For each $n \in \mathbb{N}$, we have*

$$\mathcal{N}^{(n)} \subset \mathcal{N} \cdot \text{BMOA}^{(n)} \quad \text{and} \quad (\mathcal{N}^+)^{(n)} \subset \mathcal{N}^+ \cdot \text{BMOA}^{(n)}.$$

More precisely, given $f \in \mathcal{N}$ (resp., $f \in \mathcal{N}^+$), one can find a zero-free function $g \in \mathcal{N}$ (resp., an outer function $g \in \mathcal{N}^+$) and an $h \in \text{BMOA}$ such that $f^{(n)} = gh^{(n)}$.

It should be mentioned that our method also applies to the meromorphic Nevanlinna class \mathcal{N}_{mer} , defined as the set of quotients u/v , where $u, v \in H^\infty$ and v is merely required to be non-null. Moreover, a glance at our proof of Theorem 1.1 will reveal that if the original function f is of the form F/I , with $F \in \mathcal{N}^+$ and I inner, then we may take $g = G/I^{n+1}$, with G outer. And again, just as in the H^p setting, our factorization theorem yields information on the zero sets.

Corollary 1.2. *We have*

$$\mathcal{Z}(\mathcal{N}^{(n)}) = \mathcal{Z}(\text{BMOA}^{(n)}), \quad n \in \mathbb{N}.$$

Indeed, Theorem 1.1 shows that every zero set for $\mathcal{N}^{(n)}$ is a zero set for $\text{BMOA}^{(n)}$, while the converse is immediate from the fact that $\text{BMOA} \subset \mathcal{N}$. Furthermore, since

\mathcal{N}^+ lies between BMOA and \mathcal{N} , as does every H^p with $0 < p < \infty$, Corollary 1.2 obviously implies the identity

$$\mathcal{Z}((\mathcal{N}^+)^{(n)}) = \mathcal{Z}(\text{BMOA}^{(n)})$$

and also (1.2).

Finally, restricting ourselves to the case $n = 1$, we wish to take a closer look at the inclusion

$$\mathcal{N}' \subset \mathcal{N} \cdot \text{BMOA}'$$

from Theorem 1.1, along with its \mathcal{N}^+ counterpart. We know from (1.5) that the inclusion is proper, and we now stress an important point of distinction between the two sides. Namely, the right-hand side, $\mathcal{N} \cdot \text{BMOA}'$, is ideal (i.e., invariant under multiplication by H^∞ functions), whereas the left-hand side, \mathcal{N}' , is not. Moreover, the space \mathcal{N}' is *highly nonideal* in the sense that even the identity function z is not a multiplier thereof! (Otherwise, the formula

$$g = (zg)' - zg', \quad g \in \mathcal{N},$$

would imply that \mathcal{N} is contained in \mathcal{N}' , which we know is false.) A similar remark applies to $(\mathcal{N}^+)'$.

Our last result states, then, that $\mathcal{N} \cdot \text{BMOA}'$ is actually the smallest ideal space containing \mathcal{N}' , and that the same is true in the \mathcal{N}^+ setting.

Theorem 1.3. (a) *The class $\mathcal{N} \cdot \text{BMOA}'$ is the ideal hull of \mathcal{N}' . In other words, $\mathcal{N} \cdot \text{BMOA}'$ is an ideal vector space that contains \mathcal{N}' and is contained in every ideal space X with $\mathcal{N}' \subset X$.*

(b) *Similarly, $\mathcal{N}^+ \cdot \text{BMOA}'$ is the ideal hull of $(\mathcal{N}^+)'$.*

Now let us turn to the proofs.

2. PRELIMINARIES

A couple of lemmas will be needed.

Lemma 2.1. *Let $k \geq 0$ and $l \geq 1$ be integers. If $\varphi \in \text{BMOA}^{(l)}$ and ψ is a function in $\mathcal{H}(\mathbb{D})$ satisfying*

$$(2.1) \quad \psi(z) = O((1 - |z|)^{-k}), \quad z \in \mathbb{D},$$

then $\varphi\psi \in \text{BMOA}^{(k+l)}$.

Proof. It is known (see, e.g., [8, 10, 11]) that a function $F \in \mathcal{H}(\mathbb{D})$ will be in $\text{BMOA}^{(n)}$, with $n \in \mathbb{N}$, if and only if the measure $|F(z)|^2(1 - |z|)^{2n-1}dx dy$ (where $z = x + iy$) is a Carleson measure. The required result follows from this immediately, since (2.1) yields

$$|\varphi(z)\psi(z)|^2(1 - |z|)^{2(k+l)-1} \leq \text{const} \cdot |\varphi(z)|^2(1 - |z|)^{2l-1}$$

for all $z \in \mathbb{D}$. □

When $k = 0$, the above lemma reduces to saying that

$$(2.2) \quad H^\infty \cdot \text{BMOA}^{(n)} \subset \text{BMOA}^{(n)}$$

for all $n \in \mathbb{N}$; in other words, $\text{BMOA}^{(n)}$ is an ideal space. This in turn leads to the next observation.

Lemma 2.2. *For each $n \in \mathbb{N}$, the sets $\mathcal{N} \cdot \text{BMOA}^{(n)}$ and $\mathcal{N}^+ \cdot \text{BMOA}^{(n)}$ are ideal vector spaces.*

Proof. It is clear that the two sets are invariant under multiplication by H^∞ functions, but maybe not quite obvious that they are vector spaces. It is the linearity property

$$f_1, f_2 \in \mathcal{N} \cdot \text{BMOA}^{(n)} \implies f_1 + f_2 \in \mathcal{N} \cdot \text{BMOA}^{(n)}$$

(and a similar fact with \mathcal{N}^+ in place of \mathcal{N}) that should be verified. To this end, we write

$$f_j = \frac{u_j}{v_j} \cdot w_j^{(n)} \quad (j = 1, 2),$$

where $u_j, v_j \in H^\infty$ and $w_j \in \text{BMOA}$, and where v_j is zero-free (resp., outer if the f_j 's are from $\mathcal{N}^+ \cdot \text{BMOA}^{(n)}$). Note that

$$f_1 + f_2 = \frac{1}{v_1 v_2} \cdot \left(u_1 v_2 w_1^{(n)} + u_2 v_1 w_2^{(n)} \right).$$

The two terms in brackets, and hence their sum, will be in $\text{BMOA}^{(n)}$ by virtue of (2.2), while the factor $1/(v_1 v_2)$ will be in \mathcal{N} (resp., in \mathcal{N}^+). \square

3. PROOF OF THEOREM 1.1

We treat the case of \mathcal{N} first. Take $f \in \mathcal{N}$ and write $f = u/v$, where $u, v \in H^\infty$ and v has no zeros in \mathbb{D} . We have then

$$(3.1) \quad f^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(n-k)} (1/v)^{(k)}.$$

For each $k \in \{0, \dots, n\}$, Faà di Bruno's formula (see [9, Chapter 3]) yields

$$(3.2) \quad \left(\frac{1}{v} \right)^{(k)} = \sum C(m_1, \dots, m_k) v^{-m_1 - \dots - m_k - 1} \prod_{j=1}^k (v^{(j)})^{m_j},$$

where the sum is over the k -tuples (m_1, \dots, m_k) of nonnegative integers satisfying

$$(3.3) \quad m_1 + 2m_2 + \dots + km_k = k$$

and where

$$C(m_1, \dots, m_k) = (-1)^{m_1 + \dots + m_k} \frac{(m_1 + \dots + m_k)!}{m_1! \dots m_k!} \frac{k!}{1!^{m_1} \dots k!^{m_k}}.$$

For any fixed multiindex (m_1, \dots, m_k) as above, we clearly have

$$(3.4) \quad v^{-m_1 - \dots - m_k - 1} = v^{-n-1} \cdot v^{n-m_1 - \dots - m_k},$$

the last factor on the right being bounded. Indeed,

$$(3.5) \quad v^{n-m_1-\dots-m_k} \in H^\infty,$$

since it follows from (3.3) that $n - m_1 - \dots - m_k \geq 0$. We further observe that, for $j \in \mathbb{N}$,

$$(3.6) \quad v^{(j)}(z) = O((1 - |z|)^{-j}), \quad z \in \mathbb{D}$$

(because $v \in H^\infty$), and this implies together with (3.3) that

$$(3.7) \quad \prod_{j=1}^k [v^{(j)}(z)]^{m_j} = O((1 - |z|)^{-k}), \quad z \in \mathbb{D}.$$

Combining (3.2) and (3.4), we see that the k th summand in (3.1) takes the form $v^{-n-1}w_k$, where

$$(3.8) \quad w_k := \binom{n}{k} \sum C(m_1, \dots, m_k) u^{(n-k)} v^{n-m_1-\dots-m_k} \prod_{j=1}^k (v^{(j)})^{m_j};$$

the sum is understood as in (3.2). We want to show that $w_k \in \text{BMOA}^{(n)}$, and our plan is to check the corresponding inclusion for each individual term in (3.8). Thus, we claim that the function

$$\Phi_{m_1, \dots, m_k} := u^{(n-k)} v^{n-m_1-\dots-m_k} \prod_{j=1}^k (v^{(j)})^{m_j}$$

satisfies

$$(3.9) \quad \Phi_{m_1, \dots, m_k} \in \text{BMOA}^{(n)}$$

whenever $0 \leq k \leq n$ and the m_j 's are related by (3.3).

First let us verify (3.9) in the case $k \leq n - 1$. To this end, we notice that

$$u^{(n-k)} \in (H^\infty)^{(n-k)} \subset \text{BMOA}^{(n-k)},$$

where $n - k \geq 1$, while

$$[v(z)]^{n-m_1-\dots-m_k} \prod_{j=1}^k [v^{(j)}(z)]^{m_j} = O((1 - |z|)^{-k}), \quad z \in \mathbb{D},$$

by virtue of (3.5) and (3.7). The validity of (3.9) is then guaranteed by Lemma 2.1.

Now if $k = n$, then the multiindices involved are of the form (m_1, \dots, m_n) with $\sum_{j=1}^n j m_j = n$. For any such multiindex, at least one of the m_j 's (say, m_l with an $l \in \{1, \dots, n\}$) must be nonzero, so that $m_l \geq 1$ and

$$(3.10) \quad l(m_l - 1) + \sum_{1 \leq j \leq n, j \neq l} j m_j = n - l.$$

Consider the factorization

$$\Phi_{m_1, \dots, m_n} = v^{(l)} \cdot \left\{ u v^{n-m_1-\dots-m_n} (v^{(l)})^{m_l-1} \prod_{1 \leq j \leq n, j \neq l} (v^{(j)})^{m_j} \right\}.$$

The first factor, $v^{(l)}$, is then in $(H^\infty)^{(l)}$ and hence in $\text{BMOA}^{(l)}$, while the second factor (the one in curly brackets) is $O((1 - |z|)^{-n+l})$. The latter estimate is due to (3.6) and (3.10), coupled with the fact that u and v are in H^∞ . Applying Lemma 2.1 to the current factorization, we arrive at (3.9), this time with $k = n$.

Now that (3.9) is known to be true, we infer that the functions w_k from (3.8) are all in $\text{BMOA}^{(n)}$, whence obviously $\sum_{k=0}^n w_k \in \text{BMOA}^{(n)}$. Recalling that

$$f^{(n)} = v^{-n-1} \sum_{k=0}^n w_k,$$

we finally conclude that $f^{(n)}$ can be written as $gh^{(n)}$, where $g := v^{-n-1} \in \mathcal{N}$ and h is a BMOA function satisfying $h^{(n)} = \sum_{k=0}^n w_k$.

The case of \mathcal{N}^+ is similar. This time, v is taken to be an *outer* function in H^∞ , so $g = v^{-n-1}$ will be an outer function in \mathcal{N}^+ . \square

4. PROOF OF THEOREM 1.3

We shall only prove (a), the proof of (b) being similar. We know from Lemma 2.2 that $\mathcal{N} \cdot \text{BMOA}'$ is an ideal space. Furthermore, Theorem 1.1 tells us that $\mathcal{N} \cdot \text{BMOA}'$ contains \mathcal{N}' . It remains to verify that, whenever X is an ideal space with $\mathcal{N}' \subset X$, we necessarily have

$$(4.1) \quad \mathcal{N} \cdot \text{BMOA}' \subset X.$$

Take any $g \in \mathcal{N}$ and $h \in H^\infty$. Note that

$$(4.2) \quad gh' = (gh)' - g'h,$$

where both terms on the right are in X . Indeed, $(gh)'$ is obviously in \mathcal{N}' and hence in X , while the inclusion $g'h \in X$ is due to the facts that $g' \in \mathcal{N}' \subset X$ and $hX \subset X$ (recall that X is ideal). It now follows from (4.2) that $gh' \in X$, and we have thereby checked that

$$(4.3) \quad \mathcal{N} \cdot (H^\infty)' \subset X.$$

Finally, given $\eta \in \text{BMOA}$, we invoke a result of Aleksandrov and Peller [1, Theorem 3.4] to find functions $\varphi_j, \psi_j \in H^\infty$ ($j = 1, 2$) such that $\eta' = \varphi_1\psi_1' + \varphi_2\psi_2'$. Letting $g \in \mathcal{N}$ as before, we get

$$(4.4) \quad g\eta' = g\varphi_1\psi_1' + g\varphi_2\psi_2'.$$

Here, the two terms of the form $g\varphi_j\psi_j'$ are in $\mathcal{N} \cdot (H^\infty)'$, so we infer from (4.3) that they are also in X . The right-hand side of (4.4) is therefore in X , and so is the left-hand side, $g\eta'$. Thus we conclude that $g\eta' \in X$ for all $g \in \mathcal{N}$ and $\eta \in \text{BMOA}$. This establishes (4.1) and completes the proof. \square

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